THE USE OF FINITE INTEGRAL TRANSFORMS TO SOLVE PROBLEMS OF UNSTEADY HEAT CONDUCTION IN HOLLOW CYLINDERS WITH MOVING INTERNAL BOUNDARIES

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The method of finite integral transforms is used to solve problems of unsteady heat conduction in hollow cylinders with moving internal boundaries. The relations obtained allow the temperature field and the law of motion of the boundary to be determined with the required accuracy.

Solutions have been given [1] for problems of unsteady heat conduction with a moving boundary for a semi-infinite layer and an infinite plate of thickness R with various boundary conditions at the fixed wall.

This paper describes an extension of the above method to problems of unsteady heat conduction in hollow cylinders with various boundary conditions at the free wall.

Let us examine the case with a boundary condition of the first kind at the outer wall.

Let the system of equations be:

$$\frac{\partial t}{\partial \tau} = a\left(\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\frac{\partial t}{\partial r}\right), \; R_1 < r < R_2, \; \tau > 0, \tag{1}$$

$$t(r, \tau)|_{\tau=0} = 0, \tag{2}$$

$$\left.\frac{\partial t(r, \tau)}{\partial r}\right|_{r=R_1} = -\frac{q_1(\tau)}{\lambda}, \tag{3}$$

$$t(r, \tau)|_{r=R_2} = \varphi_2(\tau). \tag{4}$$

Using the integral transform [2]

$$\bar{t}_{\mu_n}(\tau) = \int_{R_1}^{R_2} r t(r, \tau) W_0\left(\mu_n \frac{r}{R_1}\right) dr, \tag{5}$$

where

$$W_0\left(\mu_n \frac{r}{R_1}\right) = Y_1(\mu_n) J_0\left(\mu_n \frac{r}{R_1}\right) - J_1(\mu_n) Y_0\left(\mu_n \frac{r}{R_1}\right), \tag{6}$$

and μ_n are roots of the characteristic equation

$$W_0(k\mu_n) = Y_1(\mu_n) J_0(k\mu_n) - J_1(\mu_n) Y_0(k\mu_n) = 0 \tag{7}$$

for $k = R_2/R_1$, and using the conversion formula for this case in the form

$$W_0^{-1}[\bar{t}_{\mu_n}(\tau)] = t(r, \tau) = = \frac{\pi^2}{2R_1^2} \sum_{n=1}^{\infty} \frac{\mu_n^2 J_0^2(k\mu_n)\bar{t}_{\mu_n}(\tau)}{J_1^2(\mu_n) - J_0^2(k\mu_n)} W_0\left(\mu_n \frac{r}{R_1}\right), \tag{8}$$

we obtain the following solution of problem (1)-(4):

$$W_0^{-1}[\bar{t}_{\mu_n}(\tau)] = t(r, \tau) =$$

$$= \frac{\pi^2}{2R_1^2} \sum_{n=1}^{\infty} \frac{\mu_n^2 J_0^2(k\mu_n)}{J_0^2(k\mu_n) - J_1^2(\mu_n)} \exp\left[-\frac{a\mu_n^2}{R_1^2}\tau\right] \times$$

$$\times W_0\left(\mu_n \frac{r}{R_1}\right) \int_0^{\tau} \left[\frac{aR_1 q_1(\tau)}{\lambda\mu_n} + \frac{a\mu_n R_2}{R_1} \times\right. \tag{9}$$

$$\left.\times \varphi_2(\tau) W_0'\left(\mu_n \frac{R_2}{R_1}\right)\right] \exp\left[\frac{a\mu_n^2}{R_1^2}\tau\right] d\tau = f(r, \tau).$$

From (9) we can determine the time τ_0 for the temperature at the boundary $r = R_1$ to reach t_m, i. e.,

$$t_m = f_0(r, \tau_0). \tag{10}$$

The left side of (10) is the temperature distribution at time τ_0.

Taking (10) into account, we see that the termperature field at time $\tau \geq \tau_0$ will satisfy Eq. (1) $(R_1 < r < R_2, \tau > \tau_0)$, the initial condition $t(r, \tau_0) = f_0(r, \tau_0)$, condition (4) at the outer surface, and the condition at the inner surface

$$t(r, \tau)|_{r=R_1} = t_m. \tag{11}$$

From time $\tau = \tau_0$ the boundary $r = R_1$ begins to move according to the equation $r = s(\tau)$. The heat balance equation at the interface has the form

$$\lambda \frac{\partial t}{\partial r} = -\frac{Q(\tau)}{2\pi r} + \rho F \frac{ds}{d\tau}. \tag{12}$$

It is necessary to determine function $t(r, \tau)$ for $r > R_1, \tau > 0$ $(\tau_0 < \tau < T_0, T_0 = \tau_0 + T)$ and the function $s(\tau)$.

This problem may be solved using the method of [1]. As in [1], we divide T into n parts, corresponding to times $\tau_1, \tau_2, \ldots, \tau_n = T_0$, $\Delta\tau_i = \tau_{i+1} - \tau_i$. The point O_i $(r = s(\tau_i))$ on the r axis corresponds to time τ_i. We assume that O_i moves in jumps, i. e., O_i is stationary for $\tau_1 < \tau < \tau_{i+1}$, but at $\tau = \tau_{i+1}$ it jumps instantaneously to O_{i+1}. We therefore have the broken line $s_n(\tau)$ instead of $s(\tau)$.

Considering the interval $\Delta\tau_i$, we see that function t_i satisfies Eq. (1) $(r > r_i, \tau > \tau_i)$, the initial condition $t_i(r, \tau_i) = f_i(r, \tau_i) = t_{i-1}(r, \tau_i)$, the boundary conditions (11) when $r = r_i$, and (4). Applying to this problem for the interval $\Delta\tau_i$, $r \geq r_i$, the transform [2]

$$\bar{t}_{\delta_{n_i}}(\tau) = \int_{r_i}^{R_2} r t_i(r, \tau) V_0\left(\delta_{n_i} \frac{r}{r_i}\right) dr, \tag{13}$$

where

$$V_0\left(\delta_{n_i} \frac{r}{r_i}\right) = Y_0(\delta_{n_i}) J_0\left(\delta_{n_i} \frac{r}{r_i}\right) - J_0(\delta_{n_i}) Y_0\left(\delta_{n_i} \frac{r}{r_i}\right), \tag{14}$$

and δ_{n_i} are roots of the characteristic equation

$$V_0(k_i \delta_{n_i}) = Y_0(\delta_{n_i}) J_0(k_i \delta_{n_i}) - J_0(\delta_{n_i}) Y_0(k_i \delta_{n_i}) = 0 \tag{15}$$

when $k_i = R_2/r_i$, and using the conversion formula

$$V_0^{-1}[\bar{t}_{\delta_{n_i}}(\tau)] = t_i(r,\ \tau) = \\ = \frac{\pi^2}{2r_i^2}\sum_{n_i=1}^{\infty}\frac{\delta_{n_i}^2 J_0^2(k_i\delta_{n_i})\,\bar{t}_{\delta_{n_i}}(\tau)}{J_0^2(\delta_{n_i}) - J_0^2(k_i\delta_{n_i})}\,V_0\left(\delta_{n_i}\frac{r}{r_i}\right), \tag{16}$$

we obtain the solution in the form

$$V_0^{-1}[\bar{t}_{\delta_{n_i}}(\tau)] = t_i(r,\ \tau) = \\ = \frac{\pi^2}{2r_i^2}\sum_{n_i=1}^{\infty}\frac{\delta_{n_i}^2 J_0^2(k_i\delta_{n_i})\,V_0\left(\delta_{n_i}\frac{r}{r_i}\right)}{J_0^2(\delta_{n_i}) - J_0^2(k_i\delta_{n_i})}\times \\ \times\left\{a\exp\left[-\frac{a\delta_{n_i}^2}{r_i^2}\tau\right]\int_{\tau_i}^{\tau}\left[\frac{\delta_{n_i}R_2\varphi_2(\tau)}{r_i}V_0'\left(\delta_{n_i}\frac{R_2}{r_i}\right)+t_{\mathrm{m}}\right]\times\right. \\ \times\exp\left[\frac{a\delta_{n_i}^2}{r_i^2}\tau\right]d\tau+\exp\left[-\frac{a\delta_{n_i}^2}{r_i^2}(\tau-\tau_i)\right]\times \\ \left.\times\int_{r_i}^{R_2} rf_i(r,\ \tau_i)\,V_0\left(\delta_{n_i}\frac{r}{r_i}\right)dr\right\},\quad r \geqslant r_i. \tag{17}$$

Taking into account that $f_i(r,\ \tau_i) = t_{i-1}(r,\ \tau_i)$, we can express t_{i-1} in terms of t_{i-2}, using (17), and similarly express t_{i-2} in terms of t_{i-3} and so on.

Using (12), we can now determine the unknown values $r_i = s(\tau_i)$, i.e., the approximate law of motion of the boundary.

For time $\tau\,(\tau_i < \tau \leqslant \tau_{i+1})$ conditions (12) may be written in the form:

$$\frac{ds}{d\tau} = \frac{Q(\tau)}{2\pi s_i\rho F} + \frac{\lambda}{\rho F}\frac{\partial t_i}{\partial r}, \tag{18}$$

where r has been replaced by s_i on the right side; then the right side is the known function of τ. Integrating (18) with respect to τ from τ_i to τ_{i+1} and summing the equalities for $i = 0, 1, 2, \ldots, l$ $(l \leq n)$, for the function $s(\tau)$ we have

$$s(\tau_l) - s(\tau_0) = \sum_{i=0}^{l-1}[s(\tau_{i+1}) - s(\tau_i)] = \\ = \sum_{i=0}^{l-1}\int_{\tau_i}^{\tau_{i+1}}\left(\frac{Q(\tau)}{2\pi s_i\rho F} + \frac{\lambda}{\rho F}\frac{\partial t_i}{\partial r}\right)d\tau, \tag{19}$$

and, knowing $s_i(\tau)$, we can obtain $t_i(r, \tau)$ from (17) for $r \geq r_i$ and $\tau \geq \tau_i$. As regards problems with boundary conditions of the second and third kind at the outer surface, the course of the solution is no different from the foregoing.

The same integral transform formula is used to solve these problems, but with different kernels [2], and therefore different conversion formulas. For this reason we shall give only the final solution for the temperature field in the interval $\Delta\tau_i$. For problems with a boundary condition of the second kind at the outer surface we have

$$G^{-1}[\bar{t}_{\nu_{n_i}}(\tau)] = t_i(r,\ \tau) = \frac{\pi^2}{2r_i^2}\sum_{n_i=1}^{\infty}\frac{\nu_{n_i}^2 J_1^2(k_i\nu_{n_i})\,G_0\left(\nu_{n_i}\frac{r}{r_i}\right)}{J_1^2(k_i\nu_{n_i}) - J_0^2(\nu_{n_i})}\times$$

$$\times \left\{ a \exp\left[-\frac{a\,\nu_{n_i}}{r_i}\,\tau\right] \int_{\tau_i}^{\tau} \left[\frac{R_2 q_2(\tau)}{\lambda}\, G_0\left(\nu_{n_i}\,\frac{R_2}{r_i}\right) + t_{\mathrm{m}}\right] \times \right.$$

$$\times \exp\left[\frac{a\,\nu_{n_i}^2}{r_i^2}\,\tau\right] d\tau + \exp\left[-\frac{a\,\nu_{n_i}^2}{r_i^2}\,(\tau-\tau_i)\right] \times \tag{20}$$

$$\left. \times \int_{r_i}^{R_2} r f_i(r,\ \tau_i)\, G_0\left(\nu_{n_i}\,\frac{r}{r_i}\right) dr \right\},$$

where

$$G_0\left(\nu_{n_i}\,\frac{r}{r_i}\right) = Y_0(\nu_{n_i})\, J_0\left(\frac{r}{r_i}\,\nu_{n_i}\right) - J_0(\nu_{n_i}) Y_0\left(\frac{r}{r_i}\,\nu_{n_i}\right), \tag{21}$$

and ν_{n_i} are roots of the characteristic equation

$$G_1(k_i\,\nu_{n_i}) = Y_0(\nu_{n_i})\, J_1(k_i\,\nu_{n_i}) - J_0(\nu_{n_i})\, Y_1(k_i\,\nu_{n_i}) = 0. \tag{22}$$

Similarly, for problems with a boundary condition of the third kind at the outer surface

$$t_i(r,\tau) = \frac{\pi^2}{2r_i^2} \sum_{n_i=1}^{\infty} \left[\gamma_{n_i}^2 U_0\left(\gamma_{n_i}\,\frac{r}{r_i}\right)\right] \times$$

$$\times \left\{ J_0^2(\gamma_{n_i})\,[1 + k_i^2\gamma_{n_i}^2/\mathrm{Bi}_2^2] \right\}^{-1} \times$$

$$\times \left\{ \frac{t_{\mathrm{m}} r_i^2}{\gamma_{n_i}^2} \left[1 - \exp\left(-\frac{a\,\gamma_{n_i}^2}{r_i^2}\,(\tau-\tau_i)\right)\right] + \right. \tag{23}$$

$$\left. + \exp\left[-\frac{a\gamma_{n_i}^2}{r_i^2}\,(\tau-\tau_i)\right] \int_{r_i}^{R} r f_i(r,\ \tau_i)\, U_0\left(\gamma_{n_i}\frac{r}{r_i}\right) dr \right\} \times$$

$$\times \left\{ \left[J_0(k_i\,\gamma_{n_i}) - \frac{k_i\,\gamma_{n_i}}{\mathrm{Bi}_2}\, J_1(k_i\,\gamma_{n_i})\right]^{-2} - 1 \right\}^{-1},$$

where

$$\times\ U_0\left(\gamma_{n_i}\frac{r}{r_i}\right) = Y_0(\gamma_{n_i})\, J_0\left(\gamma_{n_i}\frac{r}{r_i}\right) - J_0(\gamma_{n_i})\, Y_0\left(\gamma_{n_i}\frac{r}{r_i}\right), \tag{24}$$

and γ_{n_i} are roots of the characteristic equation

$$\frac{U_0(k_i\,\gamma_{n_i})}{U_1(k_i\gamma\,_{n_i})} =$$

$$= \frac{Y_0(\gamma_{n_i})\, J_0(k_i\gamma_{n_i}) - J_0(\gamma_{n_i})\, Y_0(k_i\,\gamma_{n_i})}{Y_0(\gamma_{n_i})\, J_1(k_i\,\gamma_{n_i}) - J_0(\gamma_{n_i}) Y_1(k_i\,\gamma_{n_i})} = \frac{k_i\,\gamma_{n_i}}{\mathrm{Bi}_2}. \tag{25}$$

NOTATION

λ, a – thermal conductivity and diffusivity, respectively; t_m – melting temperature; ρ – density; F – specific heat of fusion; $Q(\tau)/2\pi r$ – specific heat flux to internal surface of hollow cylinder.

REFERENCES

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